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Folding of Digraphs

Research Article

E. M. Elkholy[1]*, A. Sakr[2] and H. Ahmed[2]

1 Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt.

2 Department of Mathematics, Faculty of Shoubra Engineering, Banha University, Banha, Egypt.

Abstract: In this paper we introduced the definition of dibipartite graphs, complete dibipartite graphs and digraph folding, then we proved that any dibipartite graph can be folded but the complete dibiparatite graph can be folded to an arc. By using adjacency matrices we described the digraph folding.

Keywords: Digraphs, dibipartite graphs, complete dibipartite graphs, folding of dibipartite graphs and adjacency matrices.

1. Introduction

A digraph D consists of a set of elements, called vertices and a list of ordered pairs of these elements,called arcs. The set of vertices is called the vertex set of D,denoted by V(D), and the list of arcs is called the arc list of D, denoted by A(D). If v and w are vertices of D, then an arc of the form vw is said to be directed from v to w [2].

A dibipartite graph is a digraph whose vertex set can be split into sets A and B in such a way that each arc (directed edge) of the digraph runs from a vertex in A to a vertex in B (or a vertex of B to a vertex of A). We can distinguish the vertices in A from those in B by drawing the former in black and the latter in white, so that each arc is incident from a black(or white) vertex to a white (or black) vertex, see Figure 1.

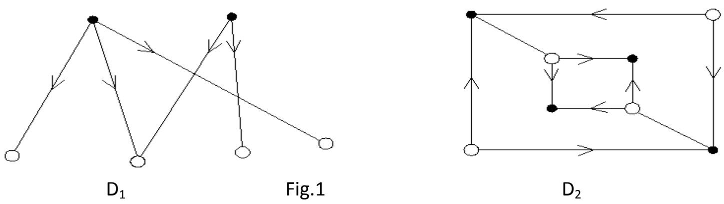

Fig.1

A complete dibipartite graph is a dibipartite graph in which each black (or white) vertex is joined to each white (or black) vertex by exactly one arc. The complete dibipartite graph with r black vertices and s white vertices is denoted by $K_{r,s}$. We call a complete dibipartite graph of the form $K_{1,s}$ star sink or star source diagraphs, see Figure 2.

Let D_1 and D_2 be digraphs and $f : D_1 \to D_2$ be a continuous function. Then f is called a digraph map if,

* *E-mail: siingam@yahoo.com*

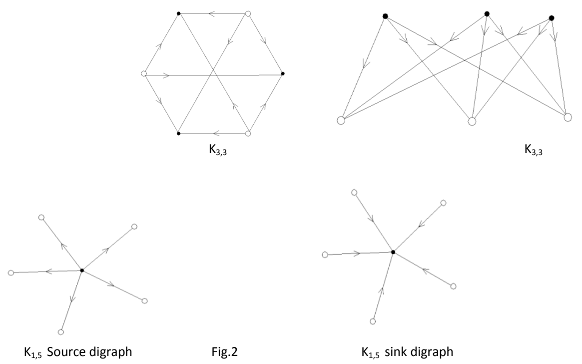

Fig.2

(1) For each vertex $v \in V(D_1)$, f(v) is a vertex in $V(D_2)$.

(2) For each arc $e \in A(D_1)$, $\dim(f(e)) \leq \dim(e)$.

2. Folding of Dibipartite graphs

Definition 2.1. *Let D_1 and D_2 be simple digraphs, we call a digraph map $f : D_1 \to D_2$ a digraph folding iff f maps vertices to vertices and arcs to arcs , i.e., for each $v \in V(D_1)$, $f(v) \in V(D_2)$ and for each $e \in A(D_1)$, $f(e) \in A(D_2)$. If the digraph contains loops, then the digraph folding must send loops to loops but of the same direction. We denote the set of diagraph foldings between diagraphs D_1 and D_2 by $\mathfrak{D}(D_1, D_2)$ and the set of diagraph foldings of D into itself by $\mathfrak{D}(D)$. In the case of a diagraph folding f the set of singularities, $\sum f$, consists of vertices only. The diagraph folding is non trivial if $\sum f \neq \phi$. In this case the no. $V(f(D_1)) \leq$ no. $V(D_1)$, also no. $A(f(D_1)) \leq$ no. $A(D_1)$.*

Example 2.2. *Let D be the diagraph shown in Figure 3. Then the graph map $f : D \to D$ defined by $f(v_1, \ldots, v_4) = (v_3, v_2, v_3, v_4)$ and $f(e_1, e_2, e_3, e_4, e_5) = (e_4, e_3, e_3, e_4, e_5)$ is a diagraph folding. The image f(D) is shown in Figure 3. From now on the omitted vertices or arcs will be mapped into themselves.*

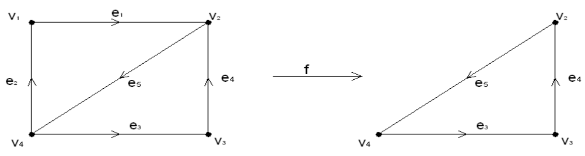

Fig.3

Theorem 2.3. *Any dibipartite graph D can be folded.*

Proof. Let D be a dibipartite graph, then the vertex set V(D) can be split into two sets A and B. Let $f : D \to D$ be a digraph map such that f maps vertices of A to vertex of A, say u, and vertices of B to a vertex of B, say v. Thus each arc e will be mapped to the arc $f(e) = (u, v)$, where $u \in V(A)$ and $v \in V(B)$ and hence f is a digraph folding . □

Example 2.4. *Let D_1 be the dibipartite graph shown in Figure 4. A digraph folding $f \in \mathfrak{D}(D_1)$ can be defined as follows $f(v_1, v_3, v_4) = (v_5, v_2, v_2)$ and $f(e_1, e_2, e_5, e_6) = (e_4, e_3, e_4, e_4)$. The image $f(D_1) = D_2$ is shown in the Figure 4.*

Theorem 2.5. *Any complete dibipartite graph D can be folded to an arc.*

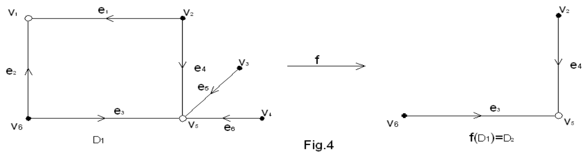

Fig.4

Proof. Let D be a complete dibipartite graph with vertex set $V(D) = \{v_1, v_2, \ldots, v_{r_1}, v_{r_1+1}, \ldots, v_r\}$. This set again can be split into two sets, $A = \{v_1, v_2, \ldots, v_{r_1}\}$ and $B = \{v_{r_1+1}, \ldots, v_r\}$ such that each vertex of A is joined to each vertex of B by exactly one arc. Thus $A(D) = \{(v_1, v_{r_1+1}), (v_1, v_{r_1+2}), \ldots, (v_1, v_r), (v_2, v_{r_2+1}), (v_2, v_{r_2+2}), \ldots, (v_2, v_r), \ldots, (v_{r_1}, v_{r_1+1}), (v_{r_1}, v_{r_2+1}), \ldots, (v_{r_1}, v_r)\}$. Now let $f : D \to D$ be a diagraph map defined by

$$f(v_k) = \begin{cases} v_1, & \text{if } k = 1, \ldots, r_1 \\ v_{r_1+1}, & \text{if } k = r_1 + 1, \ldots, r. \end{cases}$$

Thus the image of any arc of A(D) will be the arc (v_1, v_{r_1+1}). Of course, this map is a digraph folding. □

Example 2.6. *Consider the complete dibipartite graph $K_{2,4}$ shown in Figure 5. A diagraph folding f of $K_{2,4}$ into itself may be defined as follows $f(v_1, v_4, v_5, v_6) = (v_2, v_3, v_3, v_3)$, $f(e_i) = e_8$, $i = 1, \ldots, 8$. This may be done by the composition of the two digraph folding f_1 and f_2 shown in Figure 5.*

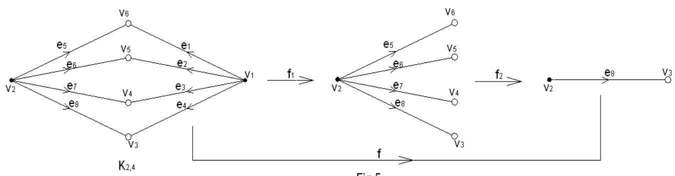

Fig.5

3. The Diagraph Folding and Adjacency Matrix

Definition 3.1. *Let D be a diagraph without loops, with n vertices labeled 1, 2, 3,..., n. The adjacency matrix M(D) is the $n \times n$ matrix in which the entry in row i and column j is the number of arcs from vertex i to vertex j [2]. For example if D is the diagraph shown in Figure 6, then the matrix M(D) will be given by*

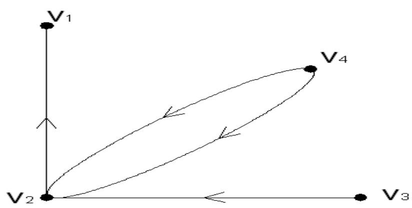

Fig.6

$$M(D) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

Note that every entry on the main diagonal (top left to bottom right) is 0, since the digraph has no loops.

Proposition 3.2. *Let D be a connected digraph without loops with n vertices. Then a digraph folding of D into itself may be defined, if there is any, as a digraph map f of D to an image f(D) by mapping:*

(i) The multiple arc into one of its arcs.

(ii) (a) The vertex v_i to the vertex v_j if the numbers appearing in the adjacency matrix in the i^{th} and j^{th} rows (or columns) are the same.

(b) The vertex v_i to the vertex v_j if the entries of the i^{th} and j^{th} rows are zeros and if the i^{th} and j^{th} columns are the same, or there exists a row k which has numbers 1 in the i^{th} and j^{th} columns.

(iii) (a) The arc (v_i, v_k) to the arc (v_j, v_k) if the i^{th} and j^{th} rows (or columns) are the same.

(b) The arc (v_i, v_j) to the arc (v_i, v_k) if the j^{th} and k^{th} columns (or rows) are the same.

In general the arc (v_i, v_j) to the arc (v_k, v_l) if v_i maps to v_k and v_j maps to v_l.

Example 3.3.

(a) Let D_1 be the digraph shown in Figure 7 The adjacency matrix $M(D_1)$ is given by

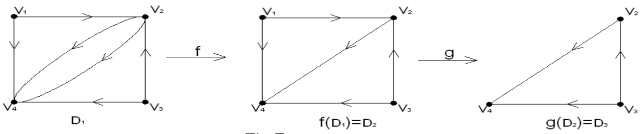

Fig.7

$$M(D_1) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Then we can fold first D_1 by folding the multiple arc into itself to get the digraph D_2. In this case $M(D_2)$ is nothing but $M(D_1)$ after replacing the number 2 by the number 1. Then a digraph folding $g \in \mathfrak{D}(D_2)$ can be defined by using $M(D_2)$ by mapping the vertex v_1 to the vertex v_3 since the first and the third row of $M(D_2)$ have the same entries. Thus the arcs (v_1, v_2) and (v_1, v_4) will be mapped to the arcs (v_3, v_2) and (v_3, v_4) respectively, since the first and the third row are the same.

(b) Let D be the digraph shown in Figure 8 The adjacency matrix $M(D)$ is given by

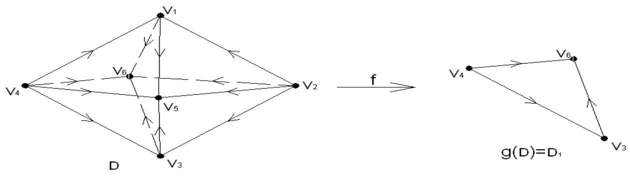

Fig.8

$$M(D) = \begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Then a digraph folding $g : D \to D$ can be defined by using M(D) by mapping the vertices v_1, v_2 and v_5 to v_3, v_4 and v_6 respectively. Also the arcs (v_2, v_1) and (v_1, v_5) will be mapped to the arcs (v_4, v_3) and (v_3, v_6) respectively since $g(v_1) = v_3$, $g(v_2) = v_4$, $g(v_5) = v_6$. Also the image of the arc (v_4, v_1) is (v_4, v_3) since the first and third columns are the same. Finally the image of the arc (v_2, v_6) is (v_4, v_6) since the second and fourth rows are the same, and so on. See the adjacency matrix M(D).

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